Math 206A Lecture 26 Notes

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1 Mechanical Linkages

1.1 Watt's linkage

Say you want to convert forces into other forces. Out story starts about 200 years ago with James Watt, who invented trains. A steam engine in a train works by having steam pressure build up and create linear motion. Watt created a mechanical linkage that (approximately) converts this linear force into a rotational force that makes the train wheels turn.

For decades, people considered the following problem: Does there exist a mechanical linkage which transfers linear into rotational motion? Chebyshev traveled to England because he was obsessed with this question. For this reason, he studied polynomials which approximate straight lines. Cayley and Sylvester were also interested in this problem. They did not believe that such a linkage could exist.

1.2 Kempe's theorem

Theorem 1.1 (A. Kempe[1], 1880s)**.** *If such a linkage exists, then for any compact portion of an algebraic curve $C \subseteq \mathbb{R}^2$, there exists a linkage with realization space $= C$.*

Think of it like this. Create a bar and joint framework for your linkage, and ground some of the vertices. Then, take a pencil at one vertex and move the framework around (which moves the pencil and draws the curve) with those grounded vertices stationary. Think of making a stencil out of your mechanical linkage.

Proof. We construct polynomials using linkages step by step:[2]

1. Rigidifying linkages: If you have two bars, linked at a joint, you can rigidify them together by adding some extra bars and joints.

2. Coordinates: If we can make lines using mechanical linkages, then we can create coordinates for $\mathbb{R}^2$ with these lines

[1] Kempe provided the first incorrect proof of the 4 color theorem.

[2] In effect, this shows how to construct mechanical computers.

3. Transfer: We can create a linkage which sets $x = y$.

4. Addition by a constant: We can create a linkage where if we know x, then we can draw $x + c$ for a constant c.

5. Multiplication by a constant: Use a linkage like a pantograph.[3]

6. Addition of vectors: If we know x, y we can construct a linkage that lets us draw $x + y$.

7. Inversion: If we have x, we can create a linkage that lets us draw $1/x$. Since the inversion of a circle is a line, and we can get both a line (by assumption) and a circle (by fixing one end of a bar, putting our pencil on the other vertex, and rotating around the fixed vertex), we can get inversions.

8. Multiplication of vectors: Note that $1/(z-1) + 1/(z+1) = 2/(z^2-1)$. So since we can do inversion, we can get squares. Then $(x+y)^2 - (x-y)^2 = 4xy$ gives us multiplication. □

There is in fact such a linkage that converts circular motion to linear motion, called the Paucillier linkage.[4] It was invented by Lipkin, but Paucillier took credit for the invention. Watt's linkage is still used in basically very car today. Why don't they use the Paucillier linkage? It has a few more moving parts, and Watt's linkage is good enough.

[3] Like me, you might be too young to know what this is.

[4] Check out some videos of it online!